ONE METHOD OF CONSTRUCTING AN OPERATIONAL CALCULUS

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A method is proposed for constructing an operational calculus, namely, by extending the concept of convolution in certain functional rings.

Let L be the set of all functions defined on the interval [0, ∞) and locally additive within this interval. The convolution of functions f, g ∈ L will be denoted as

$$f*g=\int_0^t f(t-\xi)\,g(\xi)\,d\xi. \tag{1}$$

If the product according to formula (1) is introduced into L and if addition of functions or multiplication by a number are understood in the conventional sense, then L becomes a commutative ring without divisors of zero and may be extended to a field of particular $\mathfrak{M}_S$[1].

Let ω be a linear operator defined in a linear set L_ω and covering a range of values which belong to set L.

We will postulate that

1°. $\omega f = 0$ makes $f = 0$;

2°. For all f, g ∈ L_ω there exists such an element h ∈ L_ω that

$$\omega f*\omega g=\omega h. \tag{2}$$

Since L_ω is a linear set, hence f + g and λf are defined in L_ω. If we now introduce the product for all f, g ∈ L_ω, assuming

$$fg=f\cdot g=\omega^{-1}(\omega f*\omega g)=h, \tag{3}$$

then L_ω becomes a commutative ring without divisors of zero. Let l_0 be some fixed element of ring L_ω. Let U denote a linear operator defined for all f ∈ L_ω by the condition

$$Uf=l_0 f. \tag{4}$$

Evidently, for all f, g ∈ L_ω we have

$$U(fg)=Uf\cdot g=f\cdot Ug. \tag{5}$$

Let M_0 denote the ideal in the ring L_ω generated by element l_0. The elements of ideal M_0 will be denoted by letters F, G, H, ..., and numbers will be denoted by $\lambda, \mu, \nu, \ldots$. If f ∈ M_0, then there obviously exists such an element f ∈ L_ω that

$$F=Uf. \tag{6}$$

For the elements of ideal M_0 we introduce multiplication, assuming that for all F, G ∈ M_0

$$F\otimes G=U^{-1}(FG). \tag{7}$$

The operator U^{-1} here is the reciprocal of U. Such an operator exists, since Uf = 0 makes $\omega^{-1}\cdot(\omega l_0*\omega f)=0$ and this, in turn, makes $\omega l_0*\omega f=0$. However, $\omega l_0\neq 0$ $(l_0\neq 0)$ and, therefore, $\omega f=0$

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makes $f = 0$. We will prove that $F \otimes G \in M_0$. Indeed (see (5) and (6)),

$$F \otimes G = U^{-1}(F \cdot G) = U^{-1}(Uf \cdot Ug) = f \cdot Ug = U(f \cdot g) \in M_0. \tag{8}$$

Obviously, the product is commutative, associative, and distributive. Element l_0 does not necessarily belong to ideal M_0, since ring L_ω does not necessarily include unity. For all $F \in M_0$, however, we have

$$F \otimes l_0 = U^{-1}(F \cdot l_0) = U^{-1}(Uf \cdot l_0) = f \cdot l_0 = l_0 \cdot f = Uf = F,$$
$$l_0 \otimes l_0 = U^{-1}(l_0 \cdot l_0) = U^{-1}(Ul_0) = l_0. \tag{9}$$

Thus, if a product in M_0 is understood in the sense (7) and if addition or multiplication by a number are understood as operations defined in $L_\omega \supset M_0$, then M_0 becomes a commutative ring. This ring does not necessarily include unity. If ring M_0 is extended by connecting to it all elements of the form λl_0 (λ is an arbitrary number), then element l_0 in the extended ring M is unity. We will prove that M does not have divisors of zero. Let $F + \lambda l_0 \in M$ and $G + \mu l_0 \in M$, where $F \in M_0$ and $G \in M_0$, for

$$(F + \lambda l_0) \otimes (G + \mu l_0) = 0,$$

or

$$F \otimes G + \lambda l_0 \otimes G + \mu F \otimes l_0 + \lambda\mu l_0 \otimes l_0 = 0,$$

or (see (9))

$$F \otimes G + \lambda G + \mu F + \lambda\mu l_0 = 0,$$

or (see (5), (6))

$$U(fg) + \lambda Ug + \mu Uf + \lambda\mu l_0 = 0,$$

or (see (4))

$$l_0 fg + \lambda l_0 g + \mu l_0 f + \lambda\mu l_0 = 0.$$

Multiplying both sides of the last equality by l_0, we obtain

$$l_0 f \cdot l_0 g + \lambda l_0 \cdot l_0 g + \mu l_0 \cdot l_0 f + \lambda l_0 \cdot \mu l_0 = 0,$$

which makes

$$(l_0 f + \lambda l_0)(l_0 g + \mu l_0) = 0.$$

Since ring L_ω has no divisors of zero, hence it follows from the last equality that $l_0 f + \lambda l_0 \neq 0$ if $l_0 g + \mu l_0 = 0$ and from here $G + \mu l_0 = 0$.

Thus, ring M can be extended to the field of particular $\mathfrak{M}_\omega$s. Let

$$Ul_0 = l_0 l_0 = \sigma \in M_0, \tag{10}$$

then for $F \in M_0$ we have

$$\sigma \otimes F = U^{-1}(\sigma \cdot F) = U^{-1}(Ul_0 F) = l_0 F = UF. \tag{11}$$

Consequently, operator U in ring M_0 is identical to multiplication by σ. Let

$$\frac{l_0}{\sigma} = \Omega \in \mathfrak{M}_\omega. \tag{12}$$

The $\mathfrak{M}_\omega$ field contains set L_ω. One may assume

$$\frac{F}{\sigma} = f, \quad F = Uf \in M_0.$$

In this way,

$$\sigma \otimes f = F = Uf, \quad f \in L_\omega. \tag{13}$$

For $F \in M_0$ we have

$$\Omega F = f = U^{-1} F. \tag{14}$$

We will now prove that $\mathfrak{M}_\omega$ is isomorphous with some subfield of operator $\mathfrak{M}$. Let

$$a_1 = \frac{F_1}{G_1} \in \mathfrak{M}_\omega, \quad a_2 = \frac{F_2}{G_2} \in \mathfrak{M}_\omega;$$

where F_1, G_1, F_2, and G_2 belong to M. We map $\mathfrak{M}_\omega$ into $\mathfrak{M}$, assuming for all $a = F/G \in \mathfrak{M}_\omega$

$$a \to \omega a = \omega F/\omega G \in \mathfrak{M}. \tag{15}$$

Here the operation of division in field $\mathfrak{M}$ is denoted by the slanted dash /. Let $a_1 = F_1/G_1$ and $a_2 = F_2/G_2$. Then

$$a_1 + a_2 = \frac{F_1}{G_1} + \frac{F_2}{G_2} = \frac{F_1 \otimes G_2 + F_2 \otimes G_1}{G_1 \otimes G_2}, \quad a_1 a_2 = \frac{F_1 \otimes F_2}{G_1 \otimes G_2}.$$

We will now prove that

$$\omega(a_1 + a_2) = \omega a_1 + \omega a_2, \quad \omega(a_1 a_2) = \omega a_1 * \omega a_2. \tag{16}$$

For convenience, we consider first the second of these equalities. By virtue of (11) and (5), we have

$$a_1 a_2 = \frac{F_1 \otimes F_2}{G_1 \otimes G_2} = \frac{\sigma \otimes F_1 \otimes F_2}{\sigma \otimes G_1 \otimes G_2} = \frac{UF_1 \otimes F_2}{UG_1 \otimes G_2} = \frac{U^{-1}(UF_1F_2)}{U^{-1}(UG_1G_2)} = \frac{F_1F_2}{G_1G_2}.$$

Therefore (see (3)),

$$\omega(a_1 a_2) = \omega F_1F_2/\omega G_1G_2 = \omega F_1 * \omega F_2/\omega G_1 * \omega G_2 = \omega F_1/\omega G_1 * \omega F_2/\omega G_2 = \omega a_1 * \omega a_2.$$

The first equality in (16) is proved analogously. Operator $a = F/G \in \mathfrak{M}_\omega$ will be called Laplace transformable, if operator $\omega a \in \mathfrak{M}$ is Laplace transformable. It is easy to demonstrate that the set of all Laplace transformable operators $\mathfrak{M}$ constitutes a subfield which will be denoted by $\mathfrak{N}_\omega$. To every operator $a = F/G \in \mathfrak{N}_\omega$ corresponds a function of the complex variable $\bar{a}(p)$, namely

$$a \to \bar{a}(p) = \int_0^\infty \omega F e^{-pt}\, dt \Big/ \int_0^\infty \omega G e^{-pt}\, dt. \tag{17}$$

For instance,

$$\Omega \to \omega\Omega = \omega l_0/\omega\sigma = \omega l_0/\omega l_0 * \omega l_0,$$

and if

$$\int_0^\infty \omega l_0 e^{-pt}\, dt,$$

exists, then

$$\bar{\Omega}(p) = \frac{1}{\int_0^\infty \omega l_0 e^{-pt}\, dt}. \tag{18}$$

It may happen that a product defined according to (3) is meaningful for a larger set of elements than L_ω. In this case, and if the ring is extendable, the extended ring may be considered in lieu of L_ω.

A simple illustration of the proposed theory would be the case where ω is an idem transformation ($\omega = 1$) and $l_0 = 1$.

Then

$$Uf = \int_0^t f(u)\, du, \quad Ul_0 = t, \quad U^{-1} = \frac{d}{dt},$$

M_0 is the set of all functions representable in the form

$$F(t) = \int_0^t f(u)\, du;$$

M is the set of all functions of the form

$$\int_0^t f(u)\, du + C, \; f \in L,$$

with an arbitrary constant C, and

$$F \otimes G = \frac{d}{dt} \int_0^t F(t-u)\, G(u)\, du;$$

with a classical operational calculus thus having been constructed [1].

We note, in conclusion, that $f * g$ in (1) may be regarded not only as a convolution but also any other product of functions with respect to which the original set constitutes a ring.

LITERATURE CITED

1. V. A. Ditkin and A. P. Prudnikov, Integral Transformations and Operational Calculus [in Russian], Fizmatgiz (1961).